SCATTERING BY FLAWS IN A SLAB OR A HALF-SPACE

William M. Visscher

Theoretical Division, Los Alamos National Laboratory
Los Alamos, NM 87545

I. INTRODUCTION

It is the main function of quantitative NDE to detect and to evaluate defects. Some of the most dangerous defects are cracks, especially cracks on or near surfaces. These cracks can be found by scattering ultrasonic waves from them (either bulk waves or surface waves), but up to now there is no theory (at least in the most interesting low-to-intermediate frequency region) which has been implemented to compute scattering from surface or near-surface cracks in 3d. The purpose of the present report is to explain, via a simple scalar example, the principles of a general boundary-integral-representation method which has been used[1] to calculate scattering of waves of all polarizations by a 2d surface or subsurface crack. The method is developed for bulk defects and cracks in a slab as well as in a half-space, and is straightforwardly applicable to 3-dimensional problems as well as to 2d ones.

The formalism has been developed and applied to scattering in 3d from isolated cracks.[2,3] The extension to surface cracks or cracks in slabs involves considerably more algebra in 3d than in 2d, but in principle it is straightforward manipulation.

The sections which follow include an explanation of the derivation of the Helmholtz integral equation (BIR--boundary-integral-representation) for the field in the presence of a flaw, the method used to solve it in the slab and in the half-space, and an exposition of some of the problems encountered in the numerical actualization of the solution algorithm.

The formulae directly relevant to elastic-wave scattering from surface or slab cracks are not given here, in the interest of simplicity and brevity. Our present aim is to elucidate the principles of the solution, which are in the present context the same in the scalar case elaborated here and in the elastic-wave problem to be detailed elsewhere.

II. THE INTEGRAL EQUATION

Because the formalism is simpler for the scalar case than for the vector displacement of elastodynamics, and the integral equation obtained for the latter is a straightforward generalization, we will derive the integral equation for the scalar case. One physical realization is the incompressible fluid, in which the field function is the velocity potential.

The geometry is shown in Fig. 1. A slab, bounded above by a surface S_+ and below by a surface S_-, contains a void bounded by a surface S_1. S_+ and S_- can be rough surfaces (but when the time comes actually to compute we will take them to be parallel planes).

A general linear boundary condition (b.c.) for the scalar field is, for $\vec{r}$ on S^+, S^-, or S_1,

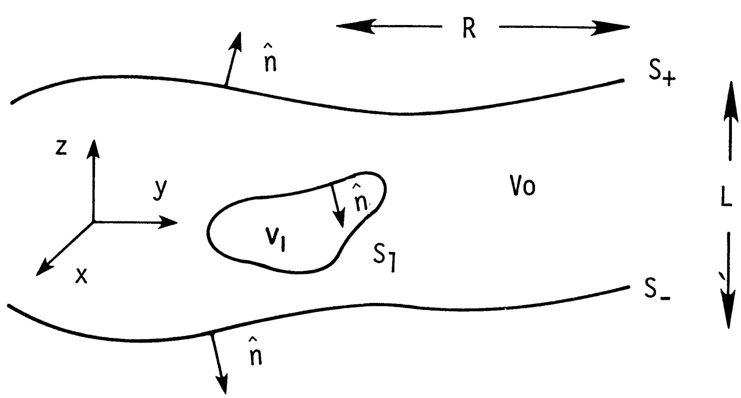

Fig. 1 The flawed slab. V_1 is a void, as is the volume above S_+ and below S_- for the slab problem. For the flawed half-space the volume below S_- is also filled with material. The effects of the edge of the slab can easily be shown to be negligible as the edge recedes ($R \to \infty$) for any finite L. $\hat{n}$ is the outward normal, as shown. The z-coordinates of points on S_+ (S_-) are finite and larger (smaller) than those of any points on S_1.

$$\hat{n} \cdot \vec{\nabla} U(\vec{r}) - \lambda U(\vec{r}) = 0 \quad , \tag{1}$$

where $\hat{n}$ is the outward-pointing unit normal. For $\lambda = 0$ this is the Neumann b.c.; for $|\lambda| = \infty$ it is the Dirichlet b.c. U(r) is, in the incompressible fluid, the velocity potential (proportional to pressure), so Neumann corresponds to the rigid wall case and Dirichlet to the free surface. The linear combination (1) is taken because it, in contrast to pure Neumann or pure Dirichlet, allows surface wave solutions which provide a scalar analog for Rayleigh waves.

The field U(r) satisfies the Helmholtz equation

$$(\nabla^2 + k^2)U(r) = 0 \tag{2}$$

in V_o. We have taken the time-dependence $\exp(-i\omega t)$ out of U, and $k = \omega/c$ where c is the sound velocity in the fluid. A Green's function can be defined

$$(\nabla^2 + k^2)G(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,') \quad ; \tag{3}$$

the solution consisting of only outgoing waves is

$$G(\vec{r},\vec{r}\,') = e^{ik|\vec{r}-\vec{r}\,'|}/4\pi|\vec{r}-\vec{r}\,'| \quad , \tag{4}$$

which can be represented in several ways as sums over orthogonal functions, corresponding to different separations, such as $r \gtrless r'$, $z \gtrless z'$, or $\rho \gtrless \rho'$ $(\rho = \sqrt{x^2 + y^2})$. The representation which will be most useful to us is[4]

$$G(\vec{r},\vec{r}\,') = 2i\int \frac{d^2q}{h(\vec{q})}\, \phi(\vec{q}_\pm,\vec{r})\phi^\dagger(\vec{q}_\pm,\vec{r}\,') \quad z \gtrless z' \quad , \tag{5}$$

where

$$\phi(\vec{q}_\pm,\vec{r}) = \frac{1}{4\pi}\, e^{i\vec{q}_\pm\cdot\vec{r}} \tag{6}$$

$$\phi^\dagger(\vec{q}_\pm,\vec{r}) = \frac{1}{4\pi}\, e^{-i\vec{q}_\pm\cdot\vec{r}} \quad , \tag{7}$$

$$\vec{q}_\pm = (q_x, q_y, \pm h(\vec{q})) \tag{8}$$

$$h(\vec{q}) = \begin{cases} \sqrt{k^2 - \vec{q}^{\,2}} & k^2 > \vec{q}^{\,2} \\ i\sqrt{\vec{q}^{\,2} - k^2} & \vec{q}^{\,2} > k^2 \end{cases} \tag{9}$$

$$\vec{q} = (q_x, q_y, 0) \quad , \tag{10}$$

and the integral is over q_x, q_y from $-\infty$ to $+\infty$.

In the usual way, we multiply (2) by G and (3) by U, subtract and integrate over V_0 to get

$$\int_{V_0} \left[G(\vec{r},\vec{r}\,')\nabla'^2 U(\vec{r}\,') - \nabla'^2 G(\vec{r},\vec{r}\,')U(\vec{r}\,') \right] d^3r' = \Theta_{V_0}(\vec{r})U(\vec{r}) \quad , \tag{11}$$

where Θ is the unit function

$$\Theta_V(\vec{r}) = \begin{cases} 1 & \vec{r} \,\varepsilon\, V \\ 0 & \text{otherwise} \end{cases} \quad . \tag{12}$$

Then, by Green's theorem, we can transform this to an integral over the surfaces bounding V_0 (see Fig. 1):

$$\Theta_{V_0}(\vec{r})U(\vec{r}) = \int_S \left[G(\vec{r},\vec{r}\,')\hat{n}'\cdot\vec{\nabla}'U(\vec{r}\,') - U(\vec{r}\,')\hat{n}'\cdot\vec{\nabla}'G(\vec{r},\vec{r}\,') \right] dS \quad , \tag{13}$$

where $S = S_+ + S_- + S_1$. (It can be easily shown that the integral over the remaining boundary does not contribute to (13) for $R \to \infty$ if the slab thickness is finite.)

It is convenient to separate out from (13) the parts which would be present even if the flaw were absent ($S_1 = 0$). So we define

$$U(\vec{r}) = U_0(\vec{r}) + U_{sc}(\vec{r}) \quad . \tag{14}$$

Here U_0 satisfies the b.c. (1) on the slab surfaces S_+ and S_-; it may consist of an incoming wave plus waves reflected back and forth between S_+ and S_-. Unless S_+ and S_- are parallel planes U_0 is usually not analytically calculable. $U_{sc}(\vec{r})$, the scattered wave, must therefore also satisfy the b.c.'s on S_+ and S_-, but only $U(\vec{r})$ satisfies the b.c.'s on S_1.

Now, it is obvious from the steps leading to (13) and the fact that U_0 satisfies Helmholtz's equation (2) in $V_0 + V_1$, that

$$\int_{S_+ + S_-} \left[G(\vec{r},\vec{r}\,')\hat{n}'\cdot\vec{\nabla}'U_0(r')-U_0(\vec{r}\,')\hat{n}'\cdot\vec{\nabla}'G(\vec{r},\vec{r}\,')\right] dS' = \Theta_{V_0+V_1}(\vec{r})U_0(\vec{r}) \ . \tag{15}$$

Therefore (13) can be written

$$\begin{aligned}\Theta_{V_0}(\vec{r})U(\vec{r}) &= \Theta_{V_0+V_1}(\vec{r})U_0(\vec{r}) \\ &+ \int_{S_++S_-}\left[G(\vec{r},\vec{r}\,')\hat{n}'\cdot\vec{\nabla}'U_{sc}(\vec{r}\,') - U_{sc}(\vec{r}\,')\hat{n}\cdot\vec{\nabla}'G(\vec{r},\vec{r}\,')\right]dS' \\ &+ \int_{S_1}\left[G(\vec{r},\vec{r}\,')\hat{n}'\cdot\vec{\nabla}'U(\vec{r}\,')-U(\vec{r}\,')\hat{n}'\cdot\vec{\nabla}'G(\vec{r},\vec{r}\,')\right]dS' \quad .\end{aligned} \tag{16}$$

This is the Helmholtz integral equation, which we will call the BIR (boundary integral representation).

Up to now we have made no direct use of the b.c.'s. Since U_{sc} satisfies the b.c.'s on S_+ and on S_- and U satisfies them on S_1, we can use the b.c.'s to eliminate the derivatives $\vec{\nabla}U$ from the integrands in (16).

Equation (16) is the integral equation appropriate to the flawed slab with a top S_+ and a bottom S_-. The flawed half-space equation can be obtained from it by relaxing the requirement that U_{sc} satisfy the b.c.'s on S_-, which is then not a physical boundary. It is a fact that

$$\begin{aligned}&\int dx'dy' \left[e^{i\vec{p}_\pm\cdot\vec{r}\,'} \frac{\partial}{\partial z'}G(\vec{r},\vec{r}\,') - G(\vec{r},\vec{r}\,')\frac{\partial}{\partial z'}e^{i\vec{p}\pm\cdot\vec{r}\,'}\right] \\ &= \left(\frac{1}{2} \pm \frac{1}{2}\right)e^{i\vec{p}_+\cdot\vec{r}} \quad z > z' \quad ,\end{aligned} \tag{17}$$

$$= \left(-\frac{1}{2} \pm \frac{1}{2}\right) e^{i\vec{p}_- \cdot \vec{r}} \quad z < z' \quad .$$

This can be readily verified by substituting (5) for G in (17). Therefore, if S_- is taken to be the x'y' plane at z' = -L (lower than S_1), for the halfspace problem the integral over S_- in (16) vanishes for z > -L because U_{sc} is composed entirely of downgoing waves, and (17) tells us that downgoing waves contribute only for z < z'.

Therefore (16) is valid for the half-space as well as for the slab geometry, with the proviso that the integration over S_- is dropped for the half-space, and U_0 need then only satisfy the boundary conditions on S_+. U_0 can be an incident surface wave or an upward incident plane wave plus waves reflected downward from S_+. It is the solution for the unflawed half-space. (It can also be a wave emanating from a local source within V_0.)

In the slab situation U_0 is, again, a solution for the unflawed slab with a source at infinity (or within V_0). It satisfies the b.c.'s on S_- as well as on S_+.

III. METHOD OF SOLUTION IN THE SLAB

The method we will use to solve Eq. (16) for $U(\vec{r})$ has three principal steps.

i) Write (16) for $\vec{r}$ above S_+ and for $\vec{r}$ below S_- (where both Θ-functions vanish), and solve the two resultant equations <u>analytically</u> for $U_{sc}(\vec{r})$ with $\vec{r}$ on S_+ and on S_-. (This is feasible, of course, only if S_+ and S_- are parallel planes, which we will take them to be.) U_{sc} will then be expressed as a linear functional of U on the flaw surface.

ii) Substitute these expressions for U_{sc} into (16) and solve for $U(\vec{r})$, $\vec{r} \in S_1$. This can be done in more than one way. One way is to consider (16) for $\vec{r} \in V_1$. Then the left-hand side vanishes and the equation involves only $U(\vec{r})$, $\vec{r} \in S_1$, which can be solved for $U(\vec{r})$ by expanding it in a set of functions on S_1. Another way is to consider (16) for $\vec{r} \in S_1$ (or an infinitesimal distance outside), use the b.c.'s to eliminate $U(\vec{r})$ from the left-hand side, and solve as before for $U(\vec{r})$, $\vec{r} \in S_1$.

iii) Once U(r) on S_1 has been obtained it can be substituted back into (16) to obtain $U(\vec{r})$ everywhere in V_0; in particular, the asymptotic expressions for $|\vec{r}| \to \infty$ yield the scattering cross-sections.

A. Eliminating $U_{sc}(\vec{r})$ on S_+ and S_-

Now take S_+ to be the $z = 0$ plane, and S_- to be the $z = -L$ plane. Then for $z > 0$ or $z < -L$ (16), with (1), becomes

$$0 = \int_{S_+ + S_-} dx'dy' U_{sc}(\vec{r}\,')(\lambda - \hat{n}\cdot\vec{\nabla}')G(\vec{r},\vec{r}\,') + \int_{S_1} U(\vec{r}\,')(\lambda - \hat{n}'\cdot\vec{\nabla}')G(\vec{r},\vec{r}\,') \quad . \tag{18}$$

On substituting (5) for $G(\vec{r},\vec{r}\,')$, using the fact that $\hat{n} = \hat{z}$ on S_+ and $\hat{n} = -\hat{z}$ on S_-, and taking the inverse Fourier transform to dispose of the d^2q integral, one obtains the following two equations;

$$(\lambda \pm ih(\vec{q})) \int_{z'=0} \phi^{\dagger}(\vec{q},\vec{r}\,')U_{sc}(\vec{r}\,')dx'dy' + (\lambda \mp ih(\vec{q}))e^{\pm ih(\vec{q})L} \int_{z'=-L} \phi^{\dagger}(\vec{q},\vec{r}\,')U_{sc}(\vec{r}\,')dx'dy' + \int_{S_1} (\lambda + i\hat{n}'\cdot\vec{q}_{\pm})\phi^{\dagger}(\vec{q}_{\pm},\vec{r}\,')U(\vec{r}\,')ds' = 0 \quad . \tag{19}$$

The upper and lower signs result from (16) for $z > 0$ and $z < -L$ respectively. The two integrals over $dx'dy'$ are just Fourier coefficients of U_{sc} on S_+ and S_-. Equation (19) comprises two equations for these two Fourier coefficients, which we label

$$F_0(\vec{q}) = \int_{z'=0} \phi^{\dagger}(\vec{q},\vec{r}\,')U_{sc}(\vec{r}\,')dx'dy'$$

$$F_{-L}(\vec{q}) = \int_{z'=-L} \phi^{\dagger}(\vec{q},\vec{r}\,')U_{sc}(\vec{r}\,')dx'dy' \quad .$$

The solution of (19) for F_0, F_{-L} is

$$F_0 = \left[(i\lambda-h)e^{-ihL}H_+-(i\lambda+h)e^{ihL}H_-\right]/D \tag{21}$$

$$F_{-L} = \left[-(i\lambda+h)H_+ + (i\lambda-h)H_-\right]/D \tag{22}$$

where D is the determinant of (19)

$$D(\vec{q}) = 2i[(\lambda^2-h^2)\sin(hL)-2\lambda h\cos(hL)] \quad , \tag{23}$$

and $H_\pm$ are the two surface integrals on S_1 which appear in (19), namely

$$H_\pm(\vec{q}) = \int_{S_1} (\lambda+i\hat{n}'\cdot\vec{q}_\pm)\phi^\dagger(\vec{q}_\pm,\vec{r}')U(\vec{r}')dS' \quad . \tag{24}$$

Equations (21) and (22) are the expressions for the scattered fields $U_{sc}(\vec{r})$ on the planes z = 0 and z = -L which we now substitute back into (16). This gives, for the field $U(\vec{r})$ for -L < z < 0,

$$\left[1-\Theta_{V_1}(\vec{r})\right]U(\vec{r}) = U_0(\vec{r}) + 2i\int\frac{d^2q}{h(\vec{q})}\left[\phi(\vec{q}_-,\vec{r})F_0(\vec{q})-\phi(\vec{q}_+,\vec{r})F_{-L}(\vec{q})\right]$$

$$+ \int_{S_1} U(\vec{r}')[\lambda-\hat{n}'\cdot\vec{\nabla}']G(\vec{r},\vec{r}')dS' \quad . \tag{25}$$

The terms on the right-hand side of (25) have obvious physical meanings. $U_0(\vec{r})$ is the incoming wave, including reflections from the confining planes. It is the solution to the problem for which the flaw S_1 is absent. The term involving $\phi(\vec{q}_-,\vec{r})$ (i.e., downgoing waves) is that part of $U_{SC}(\vec{r})$ which was most recently reflected from the z = 0 plane; that involving $\phi(\vec{q}_+,\vec{r})$ was most recently reflected by the z = -L plane. The last term is the part which was most recently scattered by the flaw S_1. One expects that this term will not contribute to the asymptotic field in the slab at $|\vec{r}| \to \infty$, or to the scattering cross-section, and, in fact, it does not.

Equation (25) is still indeterminate, however, because $F_0(\vec{q})$ and $F_{-L}(\vec{q})$ have poles on the real q axis; one at each zero of $D(\vec{q})$, Eq. (23). These zeroes, of which there are many if kL is large, define the horizontal wave-numbers q of the slab normal modes. We must specify, on physical grounds, how the poles are to be treated in the q integration.

To do this we look at the asymptotic behavior of $U_{sc}(\vec{r})$ as $|\vec{r}| \to \infty$, $-L < z < 0$. The integration $\int d^2q = \int q dq d\beta$ where β is the azimuthal angle

$$\begin{aligned} q_x &= q \cos \beta \\ q_y &= q \sin \beta \quad , \end{aligned} \tag{26}$$

and we can do the integral over β in (25) by the method of stationary phase. Namely, with (6) we have

$$2i \int \frac{d^2q}{h(q)} \phi(\vec{q}_-, \vec{r}) F_0(\vec{q}) = \frac{i}{2\pi} \int_0^\infty \frac{q^2 dq}{h(q)} \int_0^{2\pi} d\beta \, e^{i\vec{q}_- \cdot \vec{r}} F_0(\vec{q}) \quad . \tag{27}$$

When $|\vec{\rho}| \to \infty$ ($x = \rho \cos \phi$, $y = \rho \sin \phi$),

$$e^{i\vec{q}_- \cdot \vec{r}} = e^{iq\rho\cos(\phi-\beta)-ihz}$$

$$\approx \left[e^{iq\rho\left(1-\frac{(\phi-\beta)^2}{2}\right)} + e^{-iq\rho\left(1-\frac{(\phi-\beta-\pi)^2}{2}\right)} \right] e^{-ihz}, \tag{28}$$

so that (27) becomes

$$(i/2\pi\rho)^{\frac{1}{2}} \int_0^\infty \frac{q^{3/2} dq e^{-ihz}}{h(q)} \left[e^{iq\rho} F_0(q,\phi) + e^{-iq\rho} F_0(q, \pi+\phi) \right] \tag{29}$$

Thus the contributions to (25) from the reflection terms fall off asymptotically in the slab like $\rho^{-\frac{1}{2}}$. [Looking at (4) makes it clear that the contribution to (25) from the direct scattering term falls off like ρ^{-1}, therefore can be neglected.] The first term in the square bracket in (29) is an outgoing wave, the second term is an incoming wave. We know that only outgoing waves can contribute to $U_{sc}(\vec{r})$, so the contribution of the second term must be zero. It is easy to arrange that this be so. The q-integral can be expressed as a contour integral, with contours as shown in Fig. 2. If the contour is closed in the upper half-plane as shown by the solid line, then the integrals along the

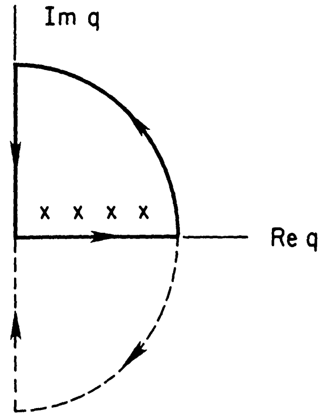

Fig. 2 Contours for the integration in Eq. (29). For $\rho \to \infty$ the integrals over the return contours vanish. If the poles of $F(\vec{q})$ are displaced into the upper half-plane as shown, then it is assured that the scattered waves will be outgoing.

quarter circle and the imaginary axis can easily be seen to vanish for the $\exp(iq\rho)$ term when $\rho \to \infty$. If it is closed in the lower half-plane as shown by the dashed lines, then the contributions of the dashed parts vanish for the $\exp(-iq\rho)$ term.

So if we displace the zeroes of $D(q)$, Eq. (23) by an infinitesimal amount into the upper half-plane, then (29) will comprise outgoing waves only, as required. The zeroes can be displaced by adding a positive imaginary infinitesimal to k^2 wherever it occurs, particularly in $h(\vec{q})$, Eq. (8). [For simplicity, we have ignored the fact that the integrand of (29) has branch points at $q = k$ and $q = 0$. Taking account of it does not alter the conclusion.]

B. Solving for $U(\vec{r})$ on S_1

Now that the ambiguities in (25) have been removed, we wish to solve it for $U(\vec{r})$ on S_1. Depending on the nature of S_1, the methods of choice can be very different. If S_1 is a surface of revolution and is fairly smooth, then a simple extension of the

T-matrix method used in Ref. 4 for the half-space problem might be used. That is, one takes $\vec{r} \in V_1$ in (25), so the left side vanishes ("extinction theorem"). One can then expand $U(\vec{r})$ on S_1, say $v_n(\vec{r})$, $n = 1,\ldots\infty$. Thus

$$U(\vec{r}) = \sum_{m=1}^{N} c_m v_m(\vec{r}) \tag{30}$$

is inserted for $U(\vec{r})$ in (25) where it occurs explicitly in the last term and in $F_0(q)$; $F_{-L}(q)$. In any practical calculation the sum in (30) must be truncated, so the question of convergence as N increases must be considered. Now (25) has become an infinite set of linear equations for c_m, $m = 1,\ldots N$; one for each point $\vec{r}$ interior to V_1. Since N is finite, an exact solution almost never exists, and some criterion must be inverted to select the "most appropriate" approximate solution.

One criterion that comes to mind is to take the integral over the volume V_1 of the absolute square of the right-hand side of (25), (which would then be bilinear in the c's) and minimize it with respect to variations in the c's. This would result in a set of N linear equations, which could then be solved for c_n. This scheme has the felicitous property (in common with MOOT) that an increase in N can be easily shown to improve the solution.

Another way to solve (25) is easier, but less aesthetic and not demonstrably convergent. This is to expand $G(\vec{r},\vec{r}\,')$, ϕ, and U_0 about a point inside S_1 in regular orthogonal functions, and match coefficients of the first N of them. This again obviously yields N equations for the N unknown c_n's, and is essentially the T-matrix method of Waterman ("extended boundary condition") applied to this problem.

The application to the crack is the one toward which the present investigation is aimed, and the above approaches are inappropriate, because $V_1 \to 0$ for the crack and variations in the last term in (25) become infinitely rapid across the crack, which makes it numerically intractable. Another method suggests itself. We operate with $(\lambda-\hat{n}\cdot\vec{\nabla})$ on (25), and take $\vec{r}$ to be on or slightly outside S_1. Then, because $U(\vec{r})$ satisfies the b.c. on S_1, we have

$$0 = (\lambda-\hat{n}\cdot\vec{\nabla})U_0(\vec{r}) + 2i\int \frac{d^2q}{h(q)}\Big[(\lambda-i\hat{n}\cdot\vec{q}_-)\phi(\vec{q}_-,\vec{r})(\lambda-ih(\vec{q}))F_0(q)$$

$$-(\lambda-i\hat{n}\cdot\vec{q}_+)\phi(\vec{q}_+,\vec{r})(\lambda+ih(\vec{q}))F_{-L}(\vec{q})\Big] + \tag{31}$$

$$+\int_{S_1} U(\vec{r}\,')(\lambda-\hat{n}\cdot\vec{\nabla})(\lambda-\hat{n}'\cdot\vec{\nabla}')G(\vec{r},\vec{r}\,')dS' \ .$$

Again, this constitutes an infinite number of equations for N c_n's if $U(\vec{r})$ is expanded as in (30) on S_1; and, again, there are many ways one might go about seeking an approximate solution. A least squares method, minimization of the integral over S_1 of the square of right-hand side of (31), is attractive but algebraically involved (especially in the elastic-wave case). The method we choose is simply to take the overlap of (31) on each of the $v_n(\vec{r})$ functions on S_1, and to solve the resultant set of N linear equations for the c's.

C. The scattered field

Once the c_n's have been obtained, $U(\vec{r})$ on S_1 can be reconstructed by (30) and substituted back into (25) to obtain $U(\vec{r})$ at all points in V_0. For points at large distances, the asymptotic scattered field is

$$U_{sc}(\vec{r}) \approx \left(\frac{i}{2\pi\rho}\right)^{\frac{1}{2}} \int_0^\infty \frac{q^{3/2}dqe^{iq\rho}}{h(q)} \left[F_0(q,\phi)e^{-ihz}-F_{-L}(q,\phi)e^{ihz}\right] , \quad (32)$$

by the argument which led to (29). The q-integral here is easy to do, because for $\rho \to \infty$ its value is just $2\pi i$ times the sum of the residues at the poles, which are located at the roots of D(q). Thus

$$U_{sc}(\rho) \approx -\sqrt{\frac{2\pi}{i\rho}} \sum_{\ell=1}^{K} \frac{q_\ell^{3/2} \ e^{iq_\ell\rho}}{h_\ell} \left[F_0(q_\ell,\phi)e^{-ih_\ell z}-F_{-L}(q_\ell,\phi)e^{ih_\ell z}\right], \quad (33)$$

where q_ℓ, $\ell = 1,\ldots K$ are the solutions of $D(q) = 0$, and $h_\ell = \sqrt{k^2-q_\ell^2}$.

IV. THE HALF-SPACE

The algebra for the half-space is simpler than for the slab. For example, although in (16) the only change is the elimination of S_- in the first surface integral, (19) is changed by the elimination of the second term and the lower signs. Thus (21) is replaced by

$$F_0(\vec{q}) = -H_+/(\lambda+ih) \quad , \qquad (34)$$

which has only one pole, corresponding to a surface wave with $q^2 = k^2 + \lambda^2$. F_{-L} is dropped from (25), and U_0 can be either a surface wave, an upward incident plane wave plus its reflected wave from S_+, or a wave from a radiating source within V_0 plus its reflections from S_+. The treatment of the surface wave pole is the same as the slab poles; it is shifted up into the complex q-plane. Now the scattered wave is not just a slab or surface wave anymore, but is the surface wave ((33) with K = 1 and without the F_{-L} term) plus a bulk wave which can be obtained from (25), again by a stationary phase argument. The reflection term (27), when both ρ and z become large ($\rho = r\sin\Theta$, $z = r\cos\Theta$, $\pi/2 < \Theta \leq \pi$) can be evaluated;

$$2i\int \frac{d^2q}{h(q)}\phi(\vec{q}_-,\vec{r})F_0(\vec{q}) \approx \frac{e^{ikr}}{r} F_0(q,\phi) \quad , \tag{35}$$

where $q = k\sin\Theta$. Thus the scattered bulk wave, in contrast to (33) for the surface wave, dies off asymptotically like 1/r; (25) becomes, for $r \to \infty$ and $\vec{r} \in V_0$;

$$U_{sc}(\vec{r}) \approx \frac{e^{ikr}}{r}\left[F_0(q,\phi) - \int_{S_1} U(\vec{r}\,')(\lambda+i\hat{n}'\cdot\vec{k})\phi^\dagger(\vec{k},\vec{r}\,')dS'\right] \quad , \tag{36}$$

where we have used (5), and $\vec{k} = k\hat{r}$. The physical interpretation here, as before, is that the first term is the part of U_{sc} which was most recently scattered by the plane surface; the second term is the part which was most recently scattered by the flaw.

Thus, for the flawed half-space, the asymptotic scattered field is obtained by first solving (31) (with $F_{-L}=0$) for $U(\vec{r})$ on S_1, then substituting the result into (33) and (36).

V. THE CRACK IN A HALF-SPACE

We will illustrate the application of the BIR method to the determination of fields scattered from a crack in a half-space. For purposes of illustration, and to simplify the algebra (at the expense of sacrificing the surface waves), we will assume that U(r) satisfies Neumann b.c.'s on S_1 and on S_+ (i.e., $\lambda=0$ in (1)). This is closely analogous to the b.c.'s satisfied by the surface traction in the elastic wave case (surface traction is a linear combination of derivatives of the displacement components). So we have

$$\hat{n}\cdot\vec{\nabla}U(r) = 0 \qquad \vec{r} \in S_1 \quad , \tag{37}$$

and (31), which we need to solve for $u(\vec{r})$, $\vec{r} \varepsilon S_1$, becomes

$$0 = -\hat{n}\cdot\vec{\nabla}U_0(\vec{r})-2i\int_{S_1} dS'U(\vec{r}')\hat{n}\cdot\vec{\nabla}\ \hat{n}'\cdot\vec{\nabla}'\int\frac{d^2q}{h(q)}\ \phi(\vec{q}_-,\vec{r})\phi^\dagger(\vec{q}_+,\vec{r}')$$

$$+\int_{S_1} dS'U(\vec{r}')\hat{n}\cdot\vec{\nabla}\ \vec{n}'\cdot\vec{\nabla}'G(\vec{r},\vec{r}') \quad . \tag{38}$$

We now let S_1 contract to a crack: the two sides become the same open surface C_1. Then the integrals over S_1 become integrals over C_1 and, because $\hat{n}'$ on one side of the crack is the negative of $\hat{n}'$ on the other side, $U(\vec{r}')$ becomes $\Delta U(\vec{r}') = U(\vec{r}'+\varepsilon\hat{n}') - U(\vec{r}'-\varepsilon\hat{n})$ (with $0 < \varepsilon << 1$), the crack opening displacement (COD). (In the elastic-wave case, also, the BIR involves only the COD's.)

The COD is now expanded according to (30), Eq. (38) is multiplied by $v_n(\vec{r})$ and is integrated over C_1. The result is

$$t_{on} + \sum_{m=1}^{N}(R_{nm} - Q_{nm})C_m = 0 \quad , \tag{39}$$

where

$$t_{on} = \int_{C_1} v_n(\vec{r})\hat{n}\cdot\vec{\nabla}U_0(\vec{r})dS \quad , \tag{40}$$

$$R_{nm} = 2i\int\frac{d^2q}{h(q)}\ \hat{n}\cdot\vec{q}_+\hat{n}\cdot\vec{q}_-\phi_n(\vec{q}_-)\phi_m^\dagger(\vec{q}_+) \quad , \tag{41}$$

$$\phi_n(\vec{q}_\pm) = \int_{C_1} dS\ v_n(\vec{r})\phi(\vec{q}_\pm,\vec{r}) \quad , \tag{42}$$

$$Q_{nm} = \int_{C_1} dS \int_{C_1} dS'\ v_n(\vec{r})v_m(\vec{r}')\hat{n}\cdot\vec{\nabla}\ \hat{n}'\cdot\nabla'G(\vec{r},\vec{r}') \quad . \tag{43}$$

Equation (39) is the set of N linear equations which determines the N unknowns C_n, n = 1,...N. It is identical to the set which determines the COD in the case of an isolated crack, except for the presence of the R (for reflection)-matrix. The Q-matrix alone determines the scattering of the isolated crack, and one can easily see that it is <u>independent</u> of the position and orientation of the crack; its elements Q_{nm} depend only on k and on $|\vec{r}_n-\vec{r}_m|$. So the Q-matrix elements can be computed in the most convenient coordinate system, namely, that in which the crack (now taken to be flat) lies in the xy plane.

$$Q_{nm} = 2i\int d^2q\ h(q)\phi_n(\vec{q})\phi_m^\dagger(\vec{q}) \quad , \tag{44}$$

and if one takes for $v_n(\vec{r})$ the normalized set of functions on the crack surface (now in the xy plane)

$$v_n(\vec{r}) = (2\pi\sigma^2)^{-1}\exp[-(\vec{r}-\vec{r}_n)^2/2\sigma^2] \quad , \tag{45}$$

then we find

$$\phi_n(\vec{q}) = \phi(\vec{q},\vec{r}_n)\exp(-\tfrac{1}{2}\sigma^2q^2) \quad , \tag{46}$$

where $\vec{r}_n$, n = 1,...N is an array of points chosen to cover the crack surface, and σ is the range of the functions $v_n(\vec{r})$. [To obtain (46) we have extended the surface integral in (42) over the whole xy plane. This was done for the sake of computational convenience; the size and shape of the crack is determined by the choice of the array $\vec{r}_n$ and by the range σ.] With (46) substituted into (44) the q-integral is obviously nicely convergent for any finite σ (the choice of σ is discussed in ref. 1), and can in some cases be performed analytically.

More troublesome is the computation of the R-matrix elements. This is because these elements do depend in an important way on the position and orientation of the crack. The reason can be seen by examining $\phi_n(\vec{q}_\pm)$ for this case:

$$\phi_n(\vec{q}_\pm) = \phi(\vec{q}_\pm,\vec{r}_n)\exp[-\tfrac{1}{2}\sigma^2(q_n^2+(q_x\cos\Psi \pm h\sin\Psi)^2)] \quad . \tag{47}$$

(The angle Ψ is as defined on Fig. 3; it is a rotation about the y-axis.) For Ψ = 0 (47) reduces to (46), up to a factor $\exp(\pm ih(q)z_n)$; if $z_n = 0$ (crack coincident with plane surface), then it is easy to see that $R_{nm} = Q_{nm}$ and there is no scattering.

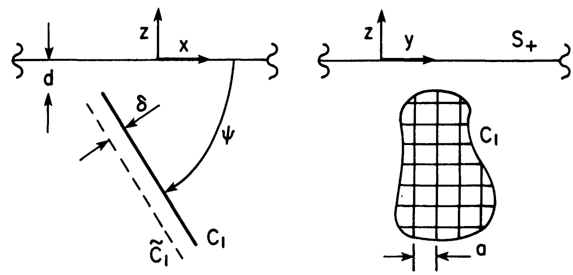

Fig. 3 The cracked half-space. The void with surface S_1 has degenerated to a flat crack, which is a flat void in which the two sides have merged to a single plane surface C_1. The crack is covered by a grid with spacing a. A localized function $v_n(\vec{r})$ defined on the crack surface is centered at each intersection.

But for $\Psi \neq 0$ (47) does not help the convergence of the integral over q in R_{nm}. In fact, for $q > k$, $h(q)$ is imaginary, so the modulus of (47) increases with increasing q, causing (41) to diverge. This behavior of $\phi_n(\vec{q}_+)$ is entirely spurious, being caused by an extension of the surface integral in (42) to the whole plane in which C_1 lies. This was done for convenience, so the surface integral could be easily done analytically. There are several ways to obviate this difficulty. One would be to restrict the integral in (42) to C_1. This would unacceptably complicate the numerical work. Another would be to mandate the convergence of (41) by summarily setting $\Psi = 0$ in the Gaussian factor in (47). This is not as unreasonable as it may at first seem, because the blowup starts only when $q\sigma \gtrsim 1$, and it makes sense to cut off the integral at wavelengths shorter than the lattice resolution; this is just what the resulting $\exp[-\frac{1}{2}\sigma^2 q^2]$ does.

A third method is the one we choose. Namely, both R_{nm} and Q_{nm} are double surface integrals over C_1; one over dS, the other over dS'. The original divergence difficulties were caused by singularities in the Green's function $G(\vec{r},\vec{r}\,')$ for $|\vec{r}-\vec{r}\,'| \to 0$. What we do to insure convergence is to displace C_1 in the direction of its normal a distance $\delta \approx a$, where a is the spacing between grid points on the crack surface (see Fig. 3). So in (38) $\vec{r}$ is not on C_1 but is on $\tilde{C}_1 = C_1 + \hat{n}\delta$, which has the effect of smearing out the singularities in the Green's function. A factor $\exp[ih(q)\delta]$ is thus introduced into the integrand of (44);

a factor $\exp[ih(q)\delta\cos\Psi + iq_x\delta\sin\Psi]$ into the integrand of (41). So (41), which already has a factor $\exp[-ih(q)(z_n + z_m)]$ in the integrand (note that $z_n < 0$) converges nicely with $\sigma \cong 0$, and, in fact, if none of the grid points are on the $z = 0$ plane, even with $\delta = 0$. And (44) converges with either or both $\sigma > 0$ and $\delta > 0$; results of computing scattering from isolated cracks are not significantly different for $\delta = 0$, $\sigma > 0$ vs. $\delta > 0$, $\sigma = 0$. Criteria for choice of δ for $\sigma = 0$ are discussed in ref. 1.

Once the c_n's have been obtained, the asymptotic fields and cross-sections can be computed by inserting (30) for the COD into appropriately modified Eqs. (33) and (36).

VI. Conclusions

In the preceding, we have derived the scalar field BIR equations for a flaw in a slab geometry or a half-space, and have explained in some detail how they may be solved for the case of a crack in a half-space. It is hoped that these relatively simple derivations will serve better to elucidate the physical, mathematical, and computational principles involved than would the more complicated expressions specific to the elastic-wave case.

The problems encountered in extending the method to the elastic-wave case are considerable, but not insuperable. Mostly they have to do with mode-conversion on reflection from the plane surfaces and from the flaw. This complicates the algebra--scalars become vectors, the Green's function becomes a dyadic and the reflection coefficient (which in the scalar case didn't need to be explicitly introduced--it was ±1) becomes a matrix function of $\vec{q}$.

The formalism has so far been numerically implemented for the case of a 2d surface or subsurface crack in an elastic medium.[3] The results will be published elsewhere. In progress is the implementation for the 3d surface or subsurface crack.

ACKNOWLEDGEMENT

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DISCUSSION

J.H. Rose (Ames Laboratory): The last couple of papers have shown that if the flaw is a significant distance below the surface, you can approximate the scattering rather simply in terms of just a mode-converted wave to the surface, and, in your case, it would be the mode-converted reflected waves. Could your formalism speak to that?

W.M. Visscher (Los Alamos National Laboratory): You mean essentially expanding powers of the reflection matrix?

J.H. Rose: Yes.

W.M. Visscher: Since the reflection matrix has these singularities I'm not sure I would trust that, but I haven't really thought about it.

J.H. Rose: For volumetric flaws it is pretty well shown. It would be interesting to see if it would do the same thing for the case of a crack.

W.M. Visscher: Well, as long as you stay away from the Rayleigh frequencies, or the Rayleigh wave numbers, I suppose it would be all right.

G.S. Kino (Stanford University): I had difficulty with some of the results that you seem to have. First of all, in regard to the Rayleigh scattering, Achenbach did calculations some time ago for a crack like that, and we did some simplified ones ourselves, and they usually show a characteristic feature as a function of depth of the crack. In other words, where it is a closed crack at the top, which is what it really is physically, it goes through a fairly sharp minimum in amplitude as a function of frequency of crack depth, KD, and then it comes up again to a maximum. When a Rayleigh wave excites it, the shear wave tends to cancel out the longitudinal wave term, and in a roundabout way, KD would equal 1. I don't see any evidence of that in your work.

W.M. Visscher: You're talking about the reflected Rayleigh wave?

G.S. Kino: Yes.

W.M. Visscher: There is some structure at the top, but you say it should go up and have a minimum and then come down?

G.S. Kino: It should have a minimum, and then begin to rise again to a maximum, and then come down, or flatten out. It doesn't seem to look the same as Achenbach's or our own simplified ones, and I think Achenbach's ones are pretty reliable. I think it's worth looking at. The other point was that when you have a shear or longitudinal wave at 45°, you should see an increase in amplitude which just varies linearly with crack depth and is much more dependent on frequency. Do you show this?

W.M. Visscher: No. That might be for one of two reasons, or at least one of at least two reasons. So far, I didn't emphasize the crudeness of this calculation. I expand the crack opening displacement on the surface of the crack in terms of just five of these localized functions. So it's not as accurate as it might be; I can make a much finer subdivision, and I will eventually. The other thing is I don't really go to very high wave numbers. I go to K equals 10. That's about 1-2/3 wavelengths on the cracked surface.

G.S. Kino: Yes, that's fine. That's the range I was talking about.

W.M. Visscher: And another thing is that it's not completely obvious to me that you should expect that, although if you say that Achenbach got it, I guess I'd better check. I haven't had time to yet. These results are only about a week old. But in the case of the SH wave, you don't get corner reflection. You can show analytically that, as a function of the angle, it's always symmetric about the down direction, about 180°. So, the SH scattering is always completely symmetric.

G. S. Kino: Well, it is a mirror; why is it a corner reflectance? There's a corner, after all?

W.M. Visscher: Yes, but in the SH case, it just doesn't work out that way. You always get exact symmetry between this direction and this direction. I don't understand why, and I don't have any intuitive feeling for why it's so, but I'm quite sure it's so.